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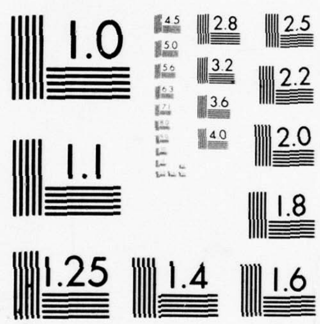

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MAXIMUM ENTROPY SPECTRAL ANALYSIS OF MULTIPLE SINUSOIDS IN NOISE

EH Satorius and JR Zeidler

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NAVAL OCEAN SYSTEMS CENTER
SAN DIEGO, CALIFORNIA 92152

NAVAL OCEAN SYSTEMS CENTER, SAN DIEGO, CA 92152

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In this paper an analytical technique based on the method of undetermined coefficients is applied to the problem of computing the theoretical maximum entropy (MEM) spectral estimate when the correlation function of the data is known exactly and corresponds to N sinusoids in additive white noise and to N sinusoids in additive 1-pole, low-pass noise. For the white noise case, the L prediction filter coefficients are expanded directly in terms of the input sinusoids. This expansion leads to a transformation of the L x L normal equations for the prediction filter coefficients to a set of 2N x 2N equations. The transformed equations are a smaller → next page

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set of equations to be solved whenever L > 2N and provide a convenient description of the interaction between the various frequency components of the sinusoids which occurs in the MEM estimate. Further, for certain cases where there is little interaction between some of the frequency components of the sinusoids, the solution of the 2N x 2N equations may be approximated (to zeroth order) by the solution of a smaller set of coupled equations. A better approximation to the exact solution of the 2N x 2N equations can then be obtained from a perturbation expansion of the exact solution about the zeroth order approximation.

For the case of N sinusoids in 1-pole, low-pass noise, the L prediction filter coefficients are expanded in terms of the input sinusoids as well as two delta functions which occur at the beginning and end of the filter. This expansion also leads to a set of 2N x 2N equations. For this case the values of the MEM estimate evaluated at the frequencies of the sinusoids are shown to be a function of the frequencies of the sinusoids. This result is reasonable since the signal-to-noise ratio per unit bandwidth is also a function of frequency.

OBJECTIVE

To examine the theoretical maximum entropy spectral estimate for multiple sinusoids in noise.

RESULTS

It was shown that for the case of sinusoids in white noise, the theoretical maximum entropy spectral estimate has very sharp peaks at the sinusoid frequencies and that the heights of these peaks are proportional to the square of the signal-to-noise ratios of the sinusoids and the square of the number of prediction filter coefficients. For the case of sinusoids in 1-pole low-pass noise, it was shown that the maximum entropy spectral resolution became dependent on the frequencies of the sinusoids. This result is reasonable, since the signal-to-noise ratio per unit bandwidth is also a function of frequency for this case.

CONTENTS

ABSTRACT

In this paper an analytical technique based on the method of undetermined coefficients is applied to the problem of computing the theoretical maximum entropy (MEM) spectral estimate when the correlation function of the data is known exactly and corresponds to N sinusoids in additive white noise and to N sinusoids in additive 1-pole, low pass noise. For the white noise case, the L prediction filter coefficients are expanded directly in terms of the input sinusoids. This expansion leads to a transformation of the L X L normal equations for the prediction filter coefficients to a set of 2N X 2N equations. The transformed equations are a smaller set of equations to be solved whenever L > 2N and provide a convenient description of the interaction between the various frequency components of the sinusoids which occurs in the MEM estimate. Further, for certain cases where there is little interaction between some of the frequency components of the sinusoids, the solution of the 2N X 2N equations may be approximated (to zeroth order) by the solution of a smaller set of coupled equations. A better approximation to the exact solution of the 2N X 2N equations can then be obtained from a perturbation expansion of the exact solution about the zeroth order approximation.

For the case of N sinusoids in 1-pole, low pass noise, the L prediction filter coefficients are expanded in terms of the input sinusoids as well as two delta functions which occur at the beginning and end of the filter. This expansion also leads to a set of 2N X 2N equations. For this case the values of the MEM estimate evaluated at the frequencies of the sinusoids are shown to be a function of the frequencies of the sinusoids. This result is reasonable since the signal-to-noise ratio per unit bandwidth is also a function of frequency.

INTRODUCTION

The maximum entropy method (MEM) of spectral analysis, which was originally proposed by Burg (1967, 1975), has been widely applied in geophysical data processing. The correspondence between MEM and linear prediction filtering, as discussed by van den Bos (1971), has allowed the application of a large body of literature on autoregressive (AR) time series analysis to MEM. Ulrych and Bishop (1975) and Ulrych and Clayton (1976) give a thorough review and discussion of MEM and AR analysis.

An important application of MEM is to the spectrum analysis of data containing multiple sinusoids in noise. Even though the proper time series model for this case is not an AR model (as will be discussed in more detail in the next sections), MEM can still provide excellent sinusoid resolution (especially for large signal-to-noise ratios). The improved resolution offered by MEM over the more conventional Fourier spectral estimation methods for this case has been well documented in the literature (e.g., Ulrych, 1972; Ulrych and Bishop, 1975). For purposes of comparing the performance of MEM with other spectral estimation techniques, Lacoss (1971) and more recently Frost (1977) and Marple (1976) have examined the theoretical MEM spectral estimate when the correlation function of the data is known exactly and consists of sinusoids in white noise. The use of an exact matrix inverse identity for computing the theoretical MEM spectral estimate for the cases of one and two sinusoids in white noise was discussed by Lacoss (1971) and Frost (1977). However, when there are more than two sinusoids, the use of the inverse identity becomes tedious. Even for two sinusoids, the analytical form of the theoretical MEM spectral estimate which is obtained from the repeated application of the inverse identity provides little insight into the interaction between the positive and negative frequency components of the sinusoids which occurs in the MEM estimate (Marple, 1976). In this paper an alternative approach based on the method of undetermined coefficients will be used to compute the theoretical MEM spectral estimate when the correlation lags of the data are known exactly and correspond to multiple sinusoids in white noise and to multiple sinusoids in 1-pole, low pass noise. It will be shown that the method of undetermined coefficients provides a convenient description of the interaction between the various frequency components of the sinusoids which occurs in the theoretical MEM estimate. Further, useful approximations for the theoretical MEM spectral estimate may be obtained for cases in which the interaction between certain sinusoidal frequency components is small but not negligible. For the case of 1-pole, low pass noise, it will be shown that the MEM spectral resolution is a function of the frequency and becomes worse in the spectral regions where the noise power increases.

MEM SPECTRAL ANALYSIS OF MULTIPLE SINUSOIDS IN WHITE NOISE

In this section we will consider the case when the correlation lags of the data, $\rho(\ell)$, $\ell=0,1,\ldots,L$, are known exactly and can be written as follows

$$\rho(\ell) = \sigma_o^2 \delta(\ell) + \sum_{n=1}^{N} \sigma_n^2 \cos 2\pi f_n \ell \tag{1}$$

where $\delta(\ell)$ is the Kronecker delta function, σ_n^2 is the power in the n^{th} sinusoid, f_n represents the frequencies of the sinusoids (which are normalized to the sample frequency), and σ_o^2 is the white noise power. The MEM spectral estimate, S(f), may be written as follows (Ülrych and Bishop, 1975)

$$S(f) = \Delta\sigma_v^2 Q(f) \tag{2}$$

where Δ is the sampling interval, and

$$Q(f) = \left|1 - \sum_{k=0}^{L-1} g(k)\, e^{-2\pi j f(k+1)}\right|^{-2} \tag{3}$$

where, f is normalized to the sample frequency. The L prediction coefficients, $g(k), k=0,\ldots,L-1$, are obtained from the normal equations

$$\sum_{k=0}^{L-1} \rho(\ell-k)\, g(k) = \rho(\ell+1), \quad \ell=0,\ldots,L-1 \tag{4}$$

and the constant σ_v^2 is obtained from the $g(k)$ by

$$\sigma_v^2 = \rho(0) - \sum_{k=0}^{L-1} g(k)\, \rho(k+1) \tag{5}$$

Lacoss (1971) and Marple (1976) treat the problem of computing $S(f)$ either through the direct numerical solution of equations (4)-(5) or through the use of a well known matrix inversion identity which is sometimes referred to as Woodbury's identity (Zielke, 1968). As noted above, the application of Woodbury's identity becomes quite tedious and leads to very extensive analytic expressions for $g(k)$ and $S(f)$ if N is larger than 2.

An alternative technique of examining $S(f)$ analytically and numerically is the method of undetermined coefficients. This method consists of substituting a solution for $g(k)$, which is expressed in terms of unknown constants, into equation (4). This substitution then leads to a set of equations for the unknown constants. This technique was originally applied to the continuous analog of equation (4) by Zadeh and Ragazzini (1950) and more recently has been applied directly to equation (4) by Satorius and Zeidler (1977) when the spectral density of the data contains both poles and zeroes.

The form of the assumed solution for $g(k)$ when $\rho(\ell)$ is given by equation (1) is

$$g(k) = \sum_{n=1}^{2N} A_n e^{2\pi j f_n k} \qquad (6)$$

where we define (for notational convenience) $f_{n+N} \equiv -f_n$ $(n=1, 2, \ldots, N)$ and the A_n are to be determined. This particular choice for the assumed solution leads to precisely $2N$ equations for the A_n. Substituting equation (6) into (4) with $\rho(\ell)$ given by equation (1) and equating coefficients of $\exp(2\pi j f_r \ell)$ (for $r=1,2,\ldots,2N$) in the resulting equation leads to the following $2N$ equations for the A_r

$$A_r + \sum_{\substack{n=1 \\ n \neq r}}^{2N} \gamma_{rn} A_n = \frac{e^{2\pi j f_r}}{L+2\sigma_o^2/\sigma_r^2}, \quad r=1,2,\ldots,2N \qquad (7)$$

where in (7) we have defined $\sigma_{n+N}^2 \equiv \sigma_n^2$ $(n=1,2,\ldots,N)$ and γ_{rn} is given by

$$\gamma_{rn} = \frac{\phi_L(f_n - f_r)}{L+2\sigma_o^2/\sigma_r^2} \qquad (8)$$

where

$$\phi_L(f) \equiv \sum_{k=0}^{L-1} e^{2\pi jfk}. \tag{9}$$

It is noted that the net effect of the method of undetermined coefficients is to transform the original $L \times L$ equations (equation (4)) to the set of $2N \times 2N$ equations (equation (7)). This transformation yields a smaller set of equations to be solved whenever $L > 2N$.

Equations (6)-(8) show that the prediction coefficients, $g(k)$, can be expressed as a sum of the positive and negative frequency components of the input sinusoids and that the amplitude of each sinusoid, A_r, is coupled to the amplitude of all the other sinusoids through coupling coefficients, γ_{rn}. The coupling coefficients vanish if $|f_n - f_r|$ is some integral multiple of $1/L$. Also, as the factor, $L + 2\sigma_0^2/\sigma_r^2$, becomes large, the γ_{rn} approach zero. As the γ_{rn} approach zero, equation (7) decouples and the A_r are given to a good approximation by

$$A_r \simeq \frac{e^{2\pi jf_r}}{L+2\sigma_o^2/\sigma_r^2}, \quad r=1,2,\ldots,2N. \tag{10}$$

Using equations (6)-(8) we may express all quantities of interest in terms of the A_n. From (5) we have for σ_v^2

$$\sigma_v^2 = \sigma_o^2 \left\{ 1 + \sum_{n=1}^{2N} A_n e^{-2\pi j f_n} \right\}. \tag{11}$$

As the γ_{rn} approach zero, σ_v^2 is given to a good approximation by

$$\sigma_v^2 \simeq \sigma_o^2 \left\{ 1 + \sum_{n=1}^{2N} \cdot \left(L + 2\sigma_0^2/\sigma_n^2 \right)^{-1} \right\}. \tag{12}$$

From equation (3), we have for $Q(f)$

$$Q(f) = \left| 1 - e^{-2\pi j f} \sum_{n=1}^{2N} A_n \phi_L(f_n - f) \right|^{-2} . \tag{13}$$

It is interesting to note that $Q(f)$ evaluated at the frequency of the r^{th} sinusoid can be simply expressed in terms of A_r and σ_o^2/σ_r^2. The result for $Q(f_r)$ is

$$Q(f_r) = |A_r|^{-2} \left(\sigma_r^4 / 4\sigma_0^4 \right), \quad r=1, \ldots, N. \tag{14}$$

Equation (14) is valid regardless of the frequency separation between the N sinusoids. (Of course, as the frequency separations approach zero, the maxima of $Q(f)$ will not necessarily be equal to the $Q(f_r)$.) As the γ_{rn} approach zero, $Q(f_r)$ is given approximately by

$$Q(f_r) \simeq \left(\sigma_r^4/4\sigma_0^4\right)\left(L + 2\sigma_o^2/\sigma_r^2\right)^2, \quad r=1,\ldots,N. \tag{15}$$

When $L \gg 2\sigma_o^2/\sigma_r^2$, equation (15) becomes

$$Q(f_r) \simeq \left(\sigma_r^4/4\sigma_o^4\right)L^2, \quad r=1,\ldots,N. \tag{16}$$

A result similar to equation (16) was also obtained by Lacoss (1971) for the theoretical peak values of the MEM estimate of a real sinusoid in white noise.

As pointed out in the recent paper by Ulrych and Clayton (1976), the proper time series model for N sinusoids in white noise is an autoregressive-moving average (ARMA) model which contains 2N autoregressive (AR) terms and 2N moving average (MA) terms. Therefore, since an infinite order AR model is required to model a finite order ARMA process (Gersch and Sharpe, 1973), it is expected that as $L \to \infty$, $S(f)$ will converge to $\sigma_o^2\Delta$ (for $f \neq f_n$; $n=1,\ldots,N$). This can be seen quite simply by noting from equation (10) that

$$\lim_{L\to\infty} A_n \phi_L (f_n - f) = 0,$$

provided $f \neq f_n$ $(n=1,\ldots,N)$. Therefore, from equations (2), (11), and (13), $S(f)$ converges pointwise to $\sigma_o^2 \Delta$ as $L \to \infty$ when $f \neq f_n$ and from equation (16) it is seen that provided the correlation lags are known exactly, the resolution capabilities of $S(f)$ inprove without limit as $L \to \infty$ regardless of the value of σ_o^2. Of course, in reality the correlation lags are never known exactly but must be estimated from a finite amount of data and, therefore, a practical limit is imposed on the value of L which provides a tradeoff between MEM resolution and the confidence in the estimates of the correlation lags (as well as the increased inaccuracies which occur when computing a larger number of prediction filter coefficients). The criteria which have been most frequently applied to the problem of determining the optimum value of L are the Akaike final prediction error (FPE) criterion (Akaike, 1969, 1970) and the Akaike information theoretic (AIC) criterion (Akaike, 1972, 1974). The application of these different criteria has been considered by a number of authors (e.g., Gersch and Sharpe, 1973; Akaike, 1974; Jones, 1976). The problem of determining the optimal value of L is further discussed by Ulrych and Bishop (1975) and Ulrych and Clayton (1976).

APPROXIMATIONS TO THE THEORETICAL MEM SPECTRAL ESTIMATE FOR SINUSOIDS IN WHITE NOISE

Note that when some of the γ_{rn} are negligible, an approximate solution (zeroth order) for the A_r may be obtained by setting the negligible γ_{rn} to zero in equation (7). A better approximation to the true A_r may be obtained by expanding equation (7) in a perturbation expansion about the zeroth order approximation. In particular, let $\underline{\underline{B}}$ be the matrix with elements B_{kj} given by

$$B_{kj} = \begin{cases} 1; \text{ if } k = j \\ \gamma_{kj}; \text{ if } \gamma_{kj} \text{ is not negligible in zeroth order} \\ 0; \text{ otherwise} \end{cases}$$

Further, let $\underline{\underline{\gamma}}^{(1)}$ be the matrix with elements $\gamma_{kj}^{(1)}$ given by

$$\gamma_{kj}^{(1)} = \begin{cases} 0; \text{ if } k = j \\ \gamma_{kj}; \text{ if } \gamma_{kj} \text{ is negligible in zeroth order} \\ 0; \text{ otherwise} \end{cases}$$

Equation (7) may now be expressed in the equivalent matrix form

$$\left(\underline{\underline{I}} + \underline{\underline{B}}^{-1} \cdot \underline{\underline{\gamma}}^{(1)}\right) \cdot \underline{A} = \underline{A}^{(0)} \tag{17}$$

where $\underline{\underline{I}}$ is the $2N \times 2N$ identity matrix; $\underline{A}$ is a column vector with k^{th} element given by A_k; and $\underline{A}^{(0)} = \underline{\underline{B}}^{-1} \cdot \underline{F}$, where $\underline{F}$ is a column vector with k^{th} element given by $\exp(2\pi j f_k)/(L + 2\sigma_o^2/\sigma_k^2)$. $\underline{A}^{(0)}$ is the zeroth order approximation to $\underline{A}$. Equation (17) can be further expanded in a perturbation expansion for $\underline{A}$ in terms of the matrix $\underline{\underline{\gamma}}^{(1)}$. The result is

$$\underline{A} = \sum_{p=0}^{\infty} \underline{A}^{(p)} \tag{18}$$

where,

$$\underline{A}^{(p)} = \left(-\underline{\underline{B}}^{-1} \cdot \underline{\underline{\gamma}}^{(1)}\right)^p \cdot \underline{A}^{(0)} . \tag{19}$$

Equation (18) is the desired perturbation expansion for $\underline{A}$ in powers of $\underline{\underline{\gamma}}^{(1)}$. The convergence of equation (18) is discussed in the Appendix.

As a specific example, consider the case of two sinusoids in white noise in which there is little interaction between the positive and negative frequency components of the sinusoids but there is appreciable interaction between the two positive frequencies (and, therefore, the two negative frequencies). For this case, the 8 coefficients γ_{13}, γ_{31}, γ_{41}, γ_{14}, γ_{23}, γ_{32}, γ_{24}, and γ_{42} may be neglected in zeroth order, and, therefore, $\underline{A}^{(0)}$ is obtained by solving the two independent sets of 2×2 equations involving only the non-negligible coefficients γ_{12}, γ_{21},

γ_{34}, and γ_{43}. From equations (18)-(19), it is straightforward to obtain approximations for Q(f). To zeroth order, Q(f) is given by

$$Q(f) \simeq Q^{(0)}(f) \equiv \left|1 - e^{-2\pi jf}\left(A_1^{(0)}\phi_L(f_1-f) + A_2^{(0)}\phi_L(f_2-f)\right)\right|^{-2}. \tag{20}$$

In equation (20), we have neglected $A_3^{(0)}$ and $A_4^{(0)}$ which is equivalent to neglecting the negative part of the frequency spectrum in Q(f) and is consistent with the neglection of the interaction between the positive and negative frequency components in the zeroth order approximation. The approximations in obtaining equation (20) are equivalent to applying Woodbury's identity to the correlation matrix formed by the positive frequency components of the two sinusoids as suggested by Lacoss (1971). Equation (20) will give a good approximation to Q(f) for f sufficiently far from zero and will also give the zeroth order approximation to $Q(f_r)$, i.e.,

$$Q^{(0)}(f_r) = \frac{1}{4}\frac{\sigma_r^4}{\sigma_o^4}\left|A_r^{(0)}\right|^{-2}, \quad r=1,2. \tag{21}$$

However, for certain applications (e.g., Frost, 1977), the errors introduced by neglecting the negative frequency components in Q(f) can be appreciable. To include these effects (to first order) we have from equations (18)-(19)

$$Q(f) \simeq Q^{(1)}(f) \equiv \left|1 - e^{-2\pi jf}\sum_{n=1}^{4}\left(A_n^{(0)} + A_n^{(1)}\right)\cdot \phi_L(f_n-f)\right|^{-2} \tag{22}$$

Equation (22) will provide a better approximation to $Q(f)$ over a wider range of frequencies than $Q^{(0)}(f)$; however, $Q^{(1)}(f_r)$ will only be correct to zeroth order, i.e., (from (19) and (22))

$$Q^{(1)}(f_r) = \frac{1}{4}\frac{\sigma_r^4}{\sigma_o^4}\left|\left(A_r^{(0)} + A_r^{(1)}\right) - \frac{\sigma_r^2}{2\sigma_o^2}\sum_{n=3}^{4} A_n^{(1)}\phi_L(f_n - f_r)\right|^{-2} \qquad (23)$$

$$r=1,2.$$

Note that for small values of σ_r^2/σ_o^2 $(r=1,2)$, $Q^{(1)}(f_r)$ will be approximately correct to first order (as can be seen from equation (23)). Higher order approximations, $Q^{(p)}(f)$ (correct to order $p-1$ at $f=f_1$ or f_2), may also be obtained as in equation (22). It should be noted that such approximations to $Q(f)$ which include the interaction between the positive and negative frequencies would be difficult to obtain using Woodbury's identity. This is because the identity would have to be applied to both the positive and negative frequencies and would result in a complicated expression for $Q(f)$ which would provide little insight into the interaction between the various frequency components of the sinusoids.

As a numerical example to illustrate the difference between the approximations for $Q(f)$ given by (20) and (22) and the exact expression given by (13), consider the case when $f_1 = .25$; $f_2 = .26$; $L = 9$; and $\sigma_1^2 = \sigma_2^2 = .1\sigma_o^2$. For this case, the magnitudes of the coupling coefficients between the positive and negative frequencies are (from (8)): $|\gamma_{24}| = |\gamma_{42}| = .029$; $|\gamma_{23}| = |\gamma_{32}| = |\gamma_{14}| = |\gamma_{41}| = .033$; $|\gamma_{13}| = |\gamma_{31}| = .034$, and the magnitude of the coupling coefficients between the closely spaced lines is: $|\gamma_{12}| = |\gamma_{21}| = |\gamma_{34}| = |\gamma_{43}| = .306$. As seen from equation (A-4), equation (18) converges for this case. In Figure 1 plots of Q(f) (exact), $Q^{(0)}(f)$, and $Q^{(1)}(f)$ are presented. As is seen, $Q^{(1)}(f)$ provides a better approximation to Q(f) than does $Q^{(0)}(f)$. It should be pointed out that the value of L used in this example (L = 9) was only chosen for purposes of comparing the different approximations for Q(f) and does not represent an optimal choice for L for this example. Indeed, as previously discussed, when the correlation function is known exactly there is no cut-off value for L and the theoretical MEM resolution improves without limit as L is increased.

MEM SPECTRAL ANALYSIS OF MULTIPLE SINUSOIDS IN 1-POLE, LOW PASS NOISE

In this section we will consider the case when the correlation lags are known exactly and can be written as

$$\rho(\ell) = \sigma_o^2 e^{-\alpha|\ell|} + \sum_{n=1}^{N} \sigma_n^2 \cos 2\pi f_n \ell \tag{24}$$

Equation (24) corresponds to the sum of N sinusoids in 1-pole, low pass noise. A determination of S(f) for this case indicates the resolution properties of MEM in a background noise with variable spectral density. For this case we expand the prediction coefficients as follows

$$g(k) = C_1\delta(k) + C_2\delta(k - L + 1) + \sum_{n=1}^{2N} A_n e^{2\pi j f_n k} . \tag{25}$$

where, as in equation (6), $f_{n+N} \equiv - f_n$ $(n=1,2,\ldots,N)$. This particular choice for the assumed solution is similar to the solution of equation (4) for the more general case when the spectral density of the data contains both poles and zeroes (Satorius and Zeidler, 1977) and leads to precisely $2N+2$ equations for the A_n, C_1, and C_2. Substituting (25) into (4) with $\rho(\ell)$ given by (24) and equating coefficients of $\exp(2\pi j f_r \ell)$, $r=1,2,\ldots,2N$, and $\exp(\pm\alpha\ell)$ in the resulting equation leads to the following $2N+2$ equations for the A_r, C_1, and C_2

$$A_r + \sum_{\substack{n=1 \\ n\neq r}}^{2N} \mu_{rn} A_n = \frac{e^{2\pi j f_r} - C_1 - C_2 e^{-2\pi j f_r (L-1)}}{L + 2\sigma_o^2/\sigma_r'^2} \tag{26}$$

$$r=1,2,\ldots,2N$$

$$C_1 = e^{-\alpha} - \sum_{n=1}^{2N} \frac{A_n}{1 - e^{\alpha+2\pi j f_n}} \tag{27}$$

$$C_2 = \sum_{n=1}^{2N} \frac{A_n e^{2\pi j f_n L}}{e^{\alpha} - e^{2\pi j f_n}} \; . \tag{28}$$

In equation (26) μ_{rn} is given by

$$\mu_{rn} = \frac{1}{L + 2\sigma_o^2/\sigma'^2_r}\, \phi_L(f_n - f_r) \tag{29}$$

and σ'^2_r is given by

$$\sigma'^2_r = \frac{1 - \cos 2\pi f_r/\cosh\alpha}{\tanh\alpha}\, \sigma^2_r \tag{30}$$

where $\sigma^2_{n+N} \equiv \sigma^2_n$ $(n=1,\ldots,N)$. Substituting equations (27)-(28) into (26) leads to the following set of $2N$ equations for the A_r

$$A_r + \sum_{\substack{n=1\\ n\neq r}}^{2N} \gamma_{rn} A_n = \left(e^{2\pi j f_r} - e^{-\alpha}\right)\beta_r \,, \qquad r=1,2,\ldots,2N \tag{31}$$

where the β_r are given by

$$\beta_r = \left\{L + 2\sigma_0^2/\sigma_r'^2 + (e^{-\alpha} - \cos 2\pi f_r)/(\cos 2\pi f_r - \cosh\alpha)\right\}^{-1} \tag{32}$$

and the γ_{rn} are given by

$$\gamma_{rn} = \left\{ \phi_L(f_n - f_r) - \left(1 - e^{\alpha + 2\pi j f_n}\right)^{-1} + \frac{e^{-\alpha + 2\pi j\left((f_n - f_r)L + f_r\right)}}{1 - e^{-\alpha + 2\pi j f_n}} \right\} \beta_r \,. \tag{33}$$

Equation (31) is similar in structure to (7) and becomes identical to (7) as $\alpha \to \infty$, as it should. Although the γ_{rn} in (33) are considerably more complicated than those for the white noise case (equation (8)), they still approach zero as $L \to \infty$. As the γ_{rn} approach zero, equation (31) decouples and the A_r are given to a good approximation by

$$A_r \simeq \left(e^{2\pi j f_r} - e^{-\alpha}\right) \beta_r . \tag{34}$$

Using equations (25)-(30) we may express all quantities of interest in terms of A_r, C_1, and C_2. From (5) we have for σ_v^2 (after considerable simplification)

$$\sigma_v^2 = \sigma_o^2\left(1-e^{-2\alpha}\right)\left\{1 + \sum_{n=1}^{2N} \frac{A_n}{e^{2\pi j f_n} - e^{-\alpha}}\right\}. \tag{35}$$

From (3) and (25), we have for $Q(f)$

$$Q(f) = \left| 1 - C_1 e^{-2\pi jf} - C_2 e^{-2\pi jfL} - e^{-2\pi jf} \sum_{n=1}^{2N} A_n \phi_L(f_n - f) \right|^{-2}. \tag{36}$$

As in the case of white noise (equation (14)), $Q(f)$ evaluated at the frequency of the r^{th} sinusoid can be simply expressed in terms of A_r and $\sigma_o^2/\sigma_r'^2$. The result for $Q(f_r)$ is

$$Q(f_r) = \frac{1}{4}\left(\frac{\sigma_r'^2}{\sigma_o^2}\right)^2 |A_r|^{-2}, \quad r=1,2,\dots,N. \tag{37}$$

As the γ_{rn} approach zero and for $L \gg \beta_r^{-1} - L$, $Q(f_r)$ is given to a good approximation by (from (34))

$$Q(f_r) \simeq \frac{e^{\alpha}}{8}\left(\frac{\sigma_r^2}{\sigma_o^2}\right)^2 L^2 \frac{\cosh\alpha - \cos 2\pi f_r}{\sinh^2\alpha}, \quad r=1,2,\dots,N. \tag{38}$$

Equation (38) shows that for large L, the values $Q(f_r)$ are a function of f_r as well as σ_r^2/σ_o^2. This result is reasonable since the signal-to-noise ratio (SNR) per unit bandwidth is a function of frequency. This indicates that the resolution capabilities of the MEM spectral estimate will be dependent on where the sinusoids are located in the noise spectrum. For sinusoids located in the low SNR regions, the MEM resolution will be worse than for sinusoids located in the high SNR region. This is indicated in Figure 2 where Q(f) is plotted for 4 sinusoids of equal amplitudes. Two of the sinusoids are located near the SNR region and the other two sinusoids are located near the high SNR region. As is seen, the two sinusoids in the low SNR region are poorly resolved whereas the two sinusoids in the high SNR region are well resolved (arrows in Figure 2 indicate the correct location of the frequencies of the sinusoids). As in the case of Figure 1, the value of L used in Figure 2 (L=14) was only chosen for purposes of illustrating the variation of two sinusoid MEM resolution versus frequency and does not represent an optimal value for L.

It should be noted that in complete analogy with the white noise case, the proper time series model for N sinusoids in additive 1-pole, low pass noise is an ARMA model which contains 2N+1 AR terms and 2N MA terms. (The additional AR term is due to the 1-pole structure of the additive noise.) Therefore, one expects that as $L \rightarrow \infty$, S(f) will converge to the power spectral density of the 1-pole noise (for $f \neq f_n$; n=1,...,N). This can easily be seen from the development presented in this section by noting from equation (34) that

$$\lim_{L\to\infty} A_n \phi_L(f_n - f) = 0,$$

provided $f \neq f_n$ $(n=1,\ldots,N)$. Further from equations (27) and (28), it is seen that $C_1 \to e^{-2}$ and $C_2 \to 0$ as $L \to \infty$. Therefore, from equations (2), (35), and (36) we have

$$\lim_{L\to\infty} S(f) = \frac{\Delta\sigma_o^2 \sinh\alpha}{\cosh\alpha - \cos 2\pi f}, \quad f \neq f_n \ (n=1,\ldots,N). \tag{39}$$

Equation (39) is the expression corresponding to the power spectral density of the 1-pole, low pass noise. Therefore, from equations (38) and (39) it is seen that S(f) provides an increasingly accurate pointwise approximation to the background noise spectrum as well as precise sinusoid resolution capabilities as $L \to \infty$. However, as noted in the previous section, this only applies when the correlation lags are known exactly.

CONCLUSIONS

In this paper, the theoretical MEM spectral estimate of multiple sinusoids in noise has been examined by the method of undetermined coefficients. For the case of sinusoids in white noise, the prediction filter coefficients were expanded directly in terms of the input sinusoids (equation (6)). This expansion leads to a transformation of the original $L \times L$ equations for the prediction coefficients (equation (4)) to a set of $2N \times 2N$ equations (equation (7)). The transformed equations are a smaller set of linear coupled equations than equation (4) when $L > 2N$ and are particularly useful for purposes of computing the theoretical MEM spectral estimate for large L (high resolution limit). Further, the reduced equations (equation (7)) have been shown to provide additional insight into the analytical structure of the MEM spectral estimate. Also, for certain cases where there is little interaction between some of the frequency components of the input sinusoids, the $2N \times 2N$ equations may be approximated by a smaller set of coupled equations.

For the case of N sinusoids in 1-pole, low pass noise, the prediction filter coefficients were expanded in terms of the input sinusoids as well as two delta functions which occur at the beginning and end of the prediction filter (equation (25)). This expansion also leads to a set of $2N \times 2N$ equations (equation

(31)). For this case, the values of the MEM spectral estimate evaluated at the input sinusoid frequencies were shown to be a function of the frequency of the sinusoids. This result is reasonable since the SNR per unit bandwidth is also a function of the frequency.

The results derived in this paper give a further understanding of the MEM spectral estimate of sinusoids in noise when the autocorrelation lags are estimated from the data. As the variance in the estimates of the lags becomes appreciable, deviations in the MEM spectral estimate will being to appear, especially at its peaks (e.g., Lacoss, 1971, Baggeroer, 1976).

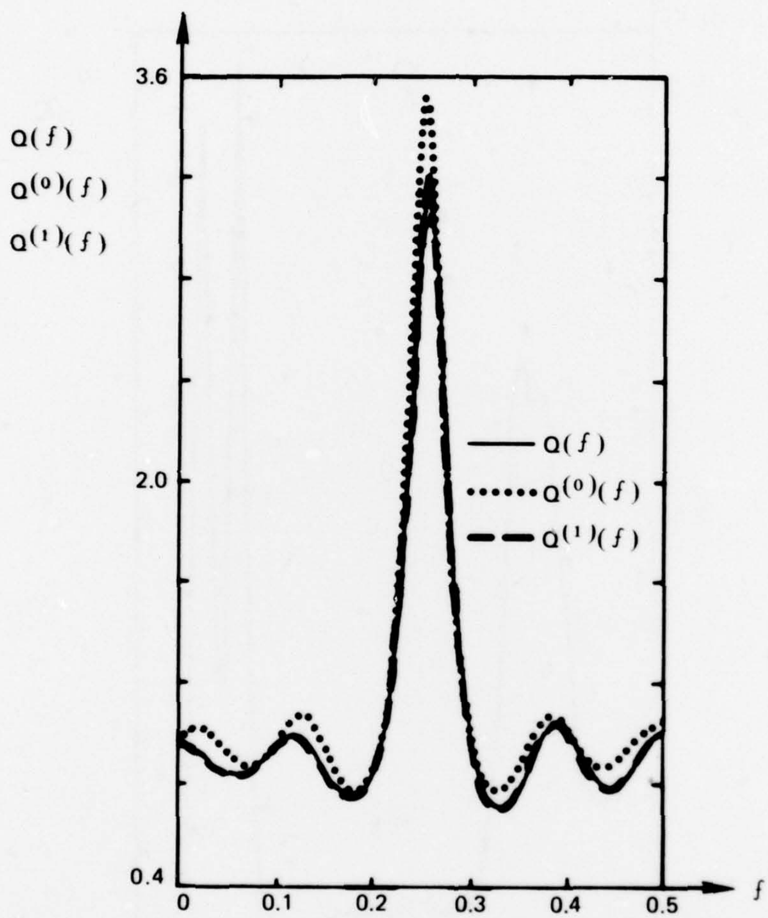

Figure 1. A comparison between the exact expression and the zeroth order and first order approximations for Q(f) for two sinusoids in white noise. For this case L = 9; $f_1 = .25$; $f_2 = .26$; and $\sigma_1^2 = \sigma_2^2 = .1\,\sigma_0^2$.

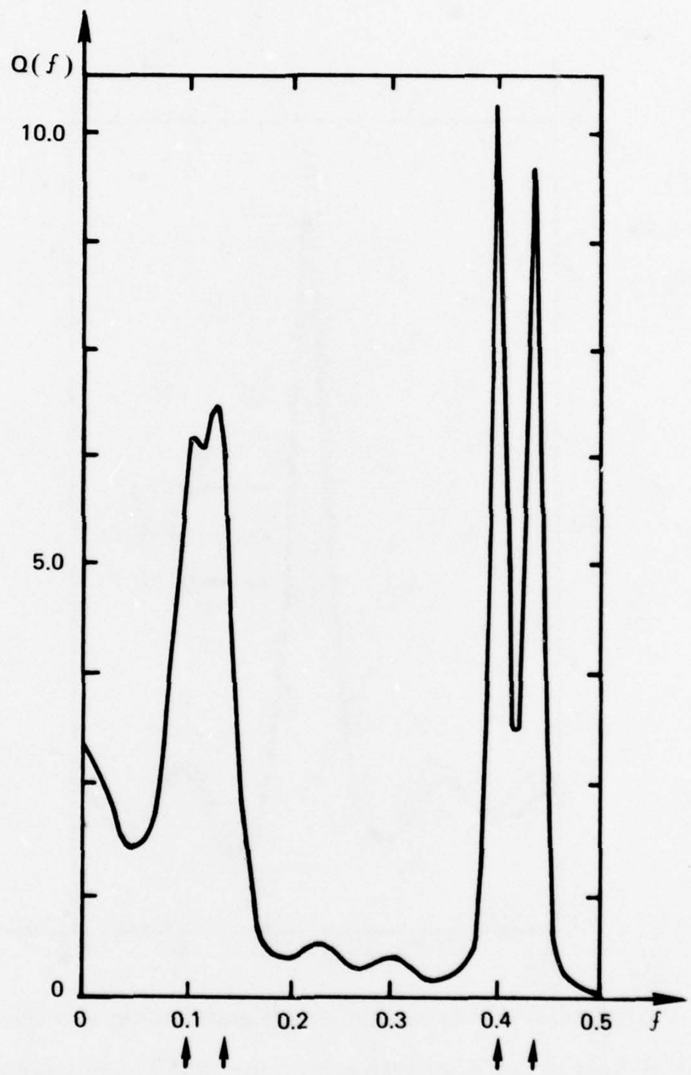

Figure 2. A plot of Q(f) for 4 sinusoids in 1 pole low pass noise. For this case L = 14; $\sigma/2\pi = .1$; $f_1 = .1$; $f_2 = .136$; $f_3 = .4$; $f_4 = .436$; and $\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = \sigma_4^2 = \sigma_0^2/2.25$.

APPENDIX A. CONDITIONS FOR CONVERGENCE OF PERTURBATION EXPANSION IN EQUATION (18)

It is the purpose of this Appendix to establish sufficient conditions for the convergence of the perturbation expansion expressed by equation (18). We make use of the following basic result (e.g., Stewart, 1973): The perturbation series $\underline{\underline{I}} + \underline{\underline{P}} + \underline{\underline{P}}^2 + \ldots$ converges to $(\underline{\underline{I}} - \underline{\underline{P}})^{-1}$ if $||\underline{\underline{P}}|| < 1$, where $||\cdot||$ denotes any matrix norm such that $||\underline{\underline{A}} \cdot \underline{\underline{C}}|| \leq ||\underline{\underline{A}}|| \; ||\underline{\underline{C}}||$.

The particular norm of which we make use and which is relatively simple to calculate is the ∞ - norm which is denoted by $||\cdot||_\infty$. The ∞ - norm of a $p \times p$ matrix, $\underline{\underline{A}}$, is defined as follows (Stewart, 1973)

$$||\underline{\underline{A}}||_\infty = \max\left(\sum_{j=1}^{p} |A_{ij}| \; : \; i=1,2,\ldots,p \right) \tag{A-1}$$

where A_{ij} are the elements of $\underline{\underline{A}}$. Therefore, a sufficient condition for the convergence of equation (18) is

$$||\underline{\underline{B}}^{-1} \cdot \underline{\underline{\gamma}}^{(1)}||_\infty < 1. \tag{A-2}$$

Since $||\underline{\underline{A}}\,\underline{\underline{C}}||_\infty \leq ||\underline{\underline{A}}||_\infty\, ||\underline{\underline{C}}||_\infty$, then a simpler sufficient condition to check is

$$||\underline{\underline{B}}^{-1}||_\infty\, ||\underline{\gamma}^{(1)}||_\infty < 1 \tag{A-3}$$

For the special case of two sinusoids of equal power in white noise in which there is little interaction between the positive and negative frequency components of the sinusoids, condition (A-3) reduces to

$$\frac{1}{1 - |\gamma_{12}|} \cdot \max \; |\gamma_{13}| + |\gamma_{14}|, \;\; |\gamma_{23}| + |\gamma_{24}| \; < 1 \tag{A-4}$$

where the γ_{ij} are computed from equation (8).

APPENDIX B. REFERENCES

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